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Comments on "Detection of a Coherent Signal of Known Phase" by R. A. Klemm

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Finding the optimum quantum detector of a coherent signal of known phase in thermal noise requires determining the spectrum of eigenvalues of the operator $\rho_1 - \lambda\rho_0$, where ρ_0 and ρ_1 are the density operators of the receiver in the absence and presence of the signal, respectively, and λ is a parameter. About the nature of the spectrum of the eigenvalues and the range of admissible values of λ the asymptotic methods used by Klemm (*Inform. Contr.* 28 (1975), 35) may convey certain misconceptions, which this note endeavors to correct.

A recent paper (Klemm, 1975) treated the quantum detection of a coherent signal of known phase in thermal noise. For sake of brevity in these comments I shall use the notation of that paper and refer to its equations by numbers preceded by the letter K. The hypotheses "signal absent" and "signal present" are as usual denoted by H_0 and H_1, respectively, and assigned prior probabilities ζ_0 and ζ_1, $\zeta_0 + \zeta_1 = 1$. For simplicity I assume the relative error costs equal, whereupon $\lambda = \zeta_0/\zeta_1$. Without loss of generality I take the signal amplitude μ to be real and positive.

Section 8 (p. 52) of Klemm's paper states, "We have shown that for the most reliable detector of a coherent signal of known phase, we must have a decision level λ less than 1." Although Section 8 does not qualify this statement, the rest of the paper might lead one to believe it is valid at least in certain asymptotic regions of the space of parameter values. Omitting the unphysical case $v_0 = 1$ (infinite temperature) and the case $\mu = 0$ (no signal), we can see by the following argument that the value of λ is in fact not restricted to the interval $0 \leqslant \lambda < 1$. However high or however low the temperature of the noise, and however weak the signal, an optimum detector must exist for values of the probability ratio λ in the range from 1 to ∞ if one exists for $0 \leqslant \lambda < 1$.

At the input to the receiver R subtract the signal to be detected, whose form is completely known. Designate what follows the subtractor S by R'. It must be possible to design this "postreceiver" R' so that the combination $S + R'$ attains the same pair (Q_0, Q_d) of false-alarm and detection probabilities as R. For R', however, hypotheses H_0 and H_1 are interchanged, and the signal amplitude μ is replaced by $-\mu$. The false-alarm and detection probabilities of

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the optimum receiver are independent of the sign of μ. Furthermore, when R' chooses H_0, $R = S + R'$ chooses H_1, and vice versa. R' can therefore be designed to choose H_1 when H_0 is true with the same probability with which R chooses H_0 when H_1 is true, and the false-alarm probability Q_0' for R' equals $1 - Q_d$. Similarly the detection probability Q_d' for R' equals $1 - Q_0$. Hence for each point (Q_0, Q_d) on the operating characteristic of the optimum receiver, there exists a point $(Q_0' = 1 - Q_d, Q_d' = 1 - Q_0)$ symmetrical with respect to the line $Q_0 + Q_d = 1$.

By a convexity argument such as was used by Holevo (1972) to derive the optimum quantum Neyman–Pearson detector, one can show that as in ordinary detection theory the slope of the operating characteristic at point (Q_0, Q_d) equals λ. For receiver R' the prior probabilities ζ_0 and ζ_1 are interchanged, and λ is replaced by $1/\lambda$. The slope at the symmetrical point (Q_0', Q_d') is therefore equal to $1/\lambda$. To $\lambda = 0$ corresponds the detection probability $Q_d = 1$, and to $\lambda = \infty$ corresponds the false-alarm probability $Q_0 = 0$. There is no value of λ in $(1, \infty)$ for which the optimum detector does not exist if a detector exists for λ in $(0, 1)$.

The structure of the optimum detector depends on the eigenvectors $|\eta\rangle$ of the operator $\rho_1 - \lambda\rho_0$, where ρ_0 and ρ_1 are the density operators of the receiver under hypotheses H_0 and H_1, respectively, and

$$(\rho_1 - \lambda\rho_0)\,|\eta_k\rangle = \eta_k\,|\eta_k\rangle. \tag{1}$$

Klemm' paper may give an erroneous impression of the spectrum of eigenvalues η_k of the operator $\rho_1 - \lambda\rho_0$. In particular, he states at the beginning of Section 7 (p. 48), "As we have shown in the previous sections, there is only one eigenvalue, except at $\lambda = 1$, where there may be two." Again excepting the point $v_0 = 1$ (infinite temperature), there are no parameter values for which this statement is precise.

Equation (K, 1.12), upon which Klemm's analysis is based, was derived by Yoshitani (1970). It was known that when $v_0 = 0$ the operator $\rho_1 - \lambda\rho_0$ possesses two nonzero eigenvalues, one of which is always positive and the other negative (Helstrom, 1968a). The associated eigenvectors span the two-dimensional subspace that is spanned by the coherent states $|0\rangle$ and $|\mu\rangle$ of the receiver under hypotheses H_0 and H_1, respectively. To the orthogonal complement of this subspace corresponds a degenerate eigenvalue $\eta = 0$. Yoshitani (1970) developed a perturbation method for calculating the eigenvalues η_k of $\rho_1 - \lambda\rho_0$ for $0 < v_0 < 1$, showing how the degenerate one at $\eta = 0$ splits, when v_0 becomes positive, into a countable infinity of eigenvalues, some of which are positive and some negative.

For $\mu > 0$ there must always be at least one positive eigenvalue and at least one negative eigenvalue of $\rho_1 - \lambda\rho_0$, as can be understood from the following argument. Let Σ_+ be the sum of the positive eigenvalues, and let Σ_- be the sum of the negative ones,

$$\Sigma_+ = \sum_k \eta_k U(\eta_k), \qquad \Sigma_- = \sum_k \eta_k U(-\eta_k), \tag{2}$$

with $U(x)$ the unit step function. Then

$$\Sigma_+ + \Sigma_- = \sum_k \eta_k = \mathrm{Tr}(\rho_1 - \lambda\rho_0) = 1 - \lambda. \tag{3}$$

The minimum average error probability attained by the optimum quantum detector is

$$P_e^m = \zeta_1(1 - \Sigma_+) = \zeta_0 + \zeta_1\Sigma_- \tag{4}$$

(Helstrom, 1968b). Consider a suboptimum detector that measures the coordinate operator Q of the modal oscillator representing the receiver, performing the optimum Bayes test on the outcome. As that outcome is a Gaussian random variable under both hypotheses, the average probability of error for the suboptimum detector is

$$P_e^\theta = \zeta_0 \operatorname{erfc}(\tfrac{1}{2}D + D^{-1}\ln\lambda) + \zeta_1 \operatorname{erfc}(\tfrac{1}{2}D - D^{-1}\ln\lambda), \tag{5}$$

where $\lambda = \zeta_0/\zeta_1$,

$$D = 2\mu\left(\frac{1 - v_0}{1 + v_0}\right)^{1/2} \tag{6}$$

is the effective signal-to-noise ratio, and

$$\operatorname{erfc} x = (2\pi)^{-(1/2)} \int_x^\infty e^{-t^2/2}\,dt$$

is the error-function integral (Helstrom *et al.*, 1970). Now for the optimum detector, by (4),

$$P_e^m = \zeta_1(1 - \Sigma_+) \leqslant P_e^\theta < \zeta_1, \qquad 0 < \zeta_1 < 1,$$

so that

$$\Sigma_+ \geqslant 1 - (P_e^\theta/\zeta_1) > 0. \tag{7}$$

Similarly

$$P_e^m = \zeta_0 + \zeta_1\Sigma_- \leqslant P_e^\theta < \zeta_0, \qquad 0 < \zeta_1 < 1,$$

and hence

$$-\Sigma_- = |\Sigma_-| \geqslant ((\zeta_0 - P_e^\theta)/\zeta_1) > 0. \tag{8}$$

Thus Σ_+ and $|\Sigma_-|$ are strictly bounded away from zero, and there must be at least one positive and one negative eigenvalue of the operator $\rho_1 - \lambda\rho_0$.

Only at the limit points $v_0 = 0$ and $v_0 = 1$ does the spectrum of eigenvalues degenerate to a finite set of numbers. As soon as the value of v_0 departs ever so little from 0 and 1, the spectrum broadens out to a countable infinity of eigenvalues. For $\mu = 0$, for instance, these eigenvalues η_k are those of $(1 - \lambda)\rho_0$, that is,

$$\eta_k = (1 - \lambda)(1 - v_0)v_0{}^k, \qquad k = 0, 1, 2,\ldots .$$

The asymptotic expansions of Sections 2, 3, 4, 5, and 6 apply to only a single one of the eigenvalues. In the analysis of Section 6, for instance, the remaining eigenvalues and their eigenvectors could be found by using, in place of (K, 6.1), series expansions for $F(v_0, \lambda \mid \alpha^*)$ beginning with each positive power of α^*, instead of with α^{*0}.

As can be seen from the coordinate representations of the density operators ρ_0 and ρ_1 (Helstrom, 1968b, Eq. (4.9)), these operators are simultaneously diagonal in that representation in the limit of infinite temperature, $v_0 \to 1$, and the eigenvectors $|\eta_k\rangle$ of $\rho_1 - \lambda\rho_0$ go into those of the coordinate operator Q. In this limit the optimum detector measures the operator Q and compares the outcome with a suitable decision level, and its average error probability is given by (5). The eigenvectors of Q, and hence those of $\rho_1 - \lambda\rho_0$ in the limit $v_0 = 1$, are not normalizable in the standard sense. It is questionable, therefore, whether the derivative $F_v(v_0, \lambda \mid \alpha^*)$ defined in (K, 2.8) exists at $v_0 = 1$ as required for Klemm's analysis.

Klemm introduces an unnecessary factor $[\operatorname{sign}(1 - \lambda)]$ into the expression (K, 5.7) for one of the two nonzero eigenvalues of $\rho_1 - \lambda\rho_0$ for $v_0 = 0$. Those two eigenvalues are

$$\begin{aligned} \eta_+ &= \tfrac{1}{2}(1 - \lambda + R) > 0, \qquad \eta_- = \tfrac{1}{2}(1 - \lambda - R) < 0, \\ R &= \{[\tfrac{1}{2}(1 - \lambda)]^2 + \lambda h\}^{1/2}, \\ h &= 1 - \exp(-\mu^2), \end{aligned} \tag{9}$$

(Helstrom, 1968a). For all values of λ there must be two distinct eigenvalues associated with two distinct eigenvectors $|\eta_+\rangle$ and $|\eta_-\rangle$, which span the two-dimensional subspace spanned by the pure coherent states $|0\rangle$ and $|\mu\rangle$. Changing the sign of R simply interchanges them and has no effect on the optimum detector or its performance.

It is vexing that the problem of finding the optimum detector of a coherent signal in thermal noise and evaluating its performance, so simple in classical detection theory, should be so difficult in quantum detection theory. The asymptotic expansions developed by Yoshitani (1970) and Klemm (1975) contribute significantly to its solution.

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